FACTORING AND OTHER TRICKS PRACTICE SHEET

[84 I-1] 1. Compute n if $(10^{12}+25)^2-(10^{12}-25)^2=10^n$

[86 I-1] 2. Compute

$$\frac{(1986^2-1992)(1986^2+3972-3)(1987)}{(1983)(1985)(1988)(1989)}$$

[80 I-1] 3. Factor completely over the set of polynomials with integral coefficients:

$$x^4-4x^3+14x^2-4x+13$$

[92 I-3] 4. Compute the largest prime factor of $3^{12}+2^{12}-2\bullet 6^6$

[92 I-1] 5. If $2(7^2+24^2)^5+3(15^2+20^2)^5=5^k$, compute k.

[89 I-4] 6. If $x=1989(a-b)$, $y=1989(b-c)$, $z=1989(c-a)$ compute the numerical value of

$$\frac{x^2+y^2+z^2}{xy+yz+xz} \quad \text{given } xy+yz+xz\neq 0$$

[85 I-5] 7. Compute the infinite sum S where

$$S=\frac{1}{2}-\frac{2}{4}+\frac{3}{8}-\frac{4}{16}+\frac{5}{32}-\ldots+\frac{n}{2^n}(-1)^{n-1}$$

[07 I-2] 8. Compute the ordered pair of positive integers (a, b) for which

$$\sqrt{151+28\sqrt{3}}=a+b\sqrt{3}$$

[04 I-4] 9. Given that $x^3+2x^2+3x+4=a(x-1)^3+b(x-1)^2+c(x-1)+d$ for all real x. Compute the ordered quadruple (a, b, c, d).

[06 I-7] 10. Compute all *real* roots of $x^2+2x-25=\sqrt{4x^2+8x-40}$

[08 I-4] 11. Compute the sum of the reciprocals of the ***real*** solutions of

$$x^5-4x^4+3x^3-8x^2+32x-24=0$$

FACTORING AND OTHER TRICKS – Solutions

1. Given a difference of squares $(10^{12}+25)^2-(10^{12}-25)^2$

$$[(10^{12}+25)+(10^{12}-25)][(10^{12}+25)-(10^{12}-25)]=$$
$$[2(10^{12})][50]=100(10^{12})=10^2(10^{12})=10^{14}$$

So $n=14$

2. Let n = 1986

$$\frac{(n^2-n-6)(n^2+2n-3)(n+1)}{(n-3)(n-1)(n+2)(n+3)}=\frac{(n-3)(n+2)(n+3)(n-1)(n+1)}{(n-3)(n-1)(n+2)(n+3)}=n+1=1987$$

3. $$\begin{aligned}x^4-4x^3+14x^2-4x+13 &= (x^4+14x^2+13)-(4x^3+4x)\\ &= (x^2+1)(x^2+13)-4x(x^2+1)\\ &= (x^2+1)(x^2-4x+13)\end{aligned}$$

4. $$\begin{aligned}3^{12}+2^{12}-2\bullet 6^6 &= 3^{12}-6^6+2^{12}-6^6\\ &= 3^6(3^6-2^6)-2^6(-2^6+3^6)=(3^6-2^6)^2\\ &= (27^2-8^2)^2=[(27-8)(27+8)]^2=[(19)(35)]^2\end{aligned}$$

Thus 19 is the largest prime factor of the given expression

5. Notice the triples $2(7^2+24^2)^5+3(15^2+20^2)^5 = 2(25^2)^5+3(25^2)^5$

$$= 2(25^{10})+3(25^{10})=25^{10}(2+3)=5^{20}(5)=5^{21}$$

So $k=21$

6. $x+y+z=1989[\,a-b+b-c+c-a\,]=0$

$$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+xz)=0$$

So $x^2+y^2+z^2=-2(xy+yz+xz)$, thus $\dfrac{x^2+y^2+z^2}{xy+yz+xz}=-2$

7. So $2S=1-\frac{2}{2}+\frac{3}{4}-\frac{4}{8}+\frac{5}{16}-\ldots$ and then adding $S+2S=3S=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$

$$3S=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots=\frac{a}{1-r}=\frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{1}{\left(\frac{3}{2}\right)}=\frac{2}{3}$$ and $S=\frac{3S}{3}=\frac{\left(\frac{2}{3}\right)}{3}=\frac{2}{9}$

8. $(a+b\sqrt{3})^2=a^2+2ab\sqrt{3}+3b^2=151+28\sqrt{3}$

So $2ab=28$, $ab=14$ and $a^2+3b^2=151$. Since $ab=14$, possible pairs are (1, 14), (2, 7), (7, 2) and (14, 1). However only (2, 7) satisfies $a^2+3b^2=2^2+3(7)^2=4+3(49)=151$. Thus the solution is (2, 7).

Solutions cont'd

9. Equate the coefficients of x^3, x^2, x and the constants and solve OR

Let $x = 1, \Rightarrow d = 10.$

Let $x = 2, \Rightarrow a + b + c + 10 = 26$

Let $x = 0, \Rightarrow -a + b - c + 10 = 4$

Now adding the last two results we get $2b = 10$ so $b = 5$, and $c = 11 - a$

Let $x = -1, \Rightarrow -8a + 20 - 2(11 - a) + 10 = 2 \Rightarrow a = 1,\ c = 10$

Thus our triple is (1, 5, 10, 10)

10. Let $y = \sqrt{x^2 + 2x - 10}$, so $y^2 - 15 = 2|y| = 2y$ as $y \geq 0$

So $y^2 - 2y - 15 = 0$ and then $(y - 5)(y + 3) = 0$, so $y = 5$ or $y = -3$, but $y \geq 0$ means $y = 5$

This gives us $5^2 = x^2 + 2x - 10$ or $x^2 + 2x - 35 = 0$,

Solving we get $(x + 7)(x - 5) = 0$ thus $x = -7$ or $x = 5$.

11. $x^5 - 4x^4 + 3x^3 - 8x^2 + 32x - 24 = 0$

$x^3(x^2 - 4x + 3) - 8(x^2 - 4x + 3) = 0$

$(x^2 - 4x + 3)(x^3 - 8) = 0$

$(x - 3)(x - 1)(x - 2)(x^2 + 2x + 4) = 0$

So $x = 3, x = 2, x = 1$ are the real solution as the solutions for $(x^2 + 2x + 4)$ are imaginary.

Thus the sum of the reciprocals of the real solutions is $\frac{1}{3} + \frac{1}{2} + 1 = \frac{2 + 3 + 6}{6} = \frac{11}{6}$

* The example problems and solutions on this practice sheet were selected from past NYSML competions and are to be used only in preparation for future NYSML or NYSML member meets. The year and question number are indicated in the brackets preceding the question.